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in: Journal of Banking and Financial Research 59 (2011) 8, p. 571

WI-225

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* The author wants to thank Dr. Wolfgang Hackenbroch for his preliminary ideas as well as Professor Dr. Hans Ulrich Buhl, Professor Dr. Andreas Rathgeber, and anonymous reviewers for their valuable comments during the preparation of this paper.

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Abstract: Striving for efficient asset allocation investors usually have limited risk bearing ability. This results in an upper limit for their portfolio volatility that must not be exceeded. Ensuring compliance with this risk limit at any point in time is essentially complicated by two factors: First, *portfolio volatility permanently changes* over time. Second, there is a *considerable time lag* between two consecutive estimations of portfolio volatility due to a restricted capacity of the employed technical resources as well as a fixed forecast horizon. This can result in a (considerable) inaccuracy with regard to future volatility which is usually accompanied either by a violation of the risk limit or by a suboptimal asset allocation yielding less than actually achievable return. In order to contribute to the prevention of these drawbacks, this paper focuses on the second factor for forecasting inaccuracy and presents the so-called Risk-at-Risk approach. This approach allows for the consideration of the time lag with respect to volatility forecasting. Its integration into asset allocation facilitates the maximisation of return while simultaneously ensuring compliance with the predefined risk limit with utmost probability. The practical applicability of the approach is demonstrated by means of two examples, one assuming future volatility to be normally or log-normally distributed and one employing real-world intra-day returns of Linde and Merck between January and September 2010. While the first example illustrates the relevance of a dynamic adjustment of forecasting frequency dependent on current market developments, the second emphasizes strong differences in asset allocation when applying Risk-at-Risk instead of mean future volatility.

Keywords: risk/return management, asset allocation, volatility forecasting, forecast horizon

JEL-Classification: D81, G32

1. Introduction

Today's investors are acting in globally integrated, extremely dynamic, and thus highly volatile market environments. They allocate their capital to risky investment opportunities like financial assets, participations, or project funding with the objective of maximising their overall return. If these investments succeed, they can achieve considerably higher rates of return than the risk-free rate. On the other hand, the failure of investments can cause serious losses and further problems as the last financial and economic crisis shows. The economically reasonable allocation of sparse capital among various assets is hence one of the fundamental problems of capital investment planning (see *Alexander* (2001), p. 186; *Bhapkar* (2007)).

Higher return is systematically associated with higher risk – as explained by classical approaches such as the Capital Asset Pricing Model (CAPM) (see e.g. *Sharpe* (1964)). Given a limited risk bearing ability, investors usually have an upper limit for their overall risk exposure. This risk limit depends on their capital bases and must not be exceeded in order to avoid the threat of liquidity problems, heavy losses or other negative consequences. An investor's investment activity and the resulting asset allocation are thus essentially governed by his risk limit.

The problem of asset allocation subject to a predefined risk limit is one of the classical issues in finance. In the context of securities, *Markowitz* (1952) was the first to answer this question by establishing the idea of maximising a portfolio's expected return subject to a risk constraint in terms of the corresponding variance. This approach was extended to a multi-period setting by *Samuelson* (1969) and to continuous time by *Merton* (1969, 1971). *Sharpe* (1964), *Lintner* (1965), and *Mossin* (1966) took it as a starting point for their well-known CAPM, which assumes investors to hold some linear combination of a fixed market portfolio and a risk-free investment alternative. Beyond this, there is recent research discussing the problem of asset allocation subject to a shortfall or Value-at-Risk (VaR) constraint (see e.g. *Campbell et al.* (2001), *Cuoco et al.* (2008), *Daníelsson et al.* (2008), *Hainaut* (2009), *Jansen et al.* (2000), or *Yin* (2004)) as well as the challenge of dealing with illiquid assets (see e.g. *Budhraja/de Figueiredo* (2005), *Cao/Teïletche* (2007), *Gârleanu* (2009), *Koren/Szeidl* (2002), *Longstaff* (2001), or *Milevsky* (2004)), which makes the problem more difficult as trading cannot "take place at short notice in large quantities without a substantial price change" (*Koren/Szeidl* (2002), p. 1).

Apart from the specific characteristics of different risk measures or asset classes, ensuring compliance with a predefined risk limit at any point in time is fundamentally complicated by two factors: First, *portfolio risk permanently changes* over time, either willingly (by making investment decisions) or unwillingly (by movement of the underlying markets). Second, there can be *a considerable time lag* between two consecutive estimations of portfolio risk due to a restricted capacity of the employed technical resources as well as a (considerable) forecast horizon. Accordingly, information about the risk currently being taken might be inaccurate as well as significantly outdated. As a consequence, the risk limit could temporarily be exceeded without the investor noticing it. In order to avoid the unnoticed violation of the predefined risk limit and the resulting negative consequences, in practice investors establish capital buffers, i.e. they build higher risk-freely invested capital reserves than actually necessary. This behaviour, however, usually leads to an economically unfavourable asset allocation resulting in radical profit cuts. Ensuring a continuous re-estimation of portfolio risk and a subsequent asset re-allocation, which aims at maximising overall return while simultaneously satisfying a predefined risk limit, is thus a crucial task of risk/return management. It is generally relevant for all sorts of industries, regardless of whether they invest in financial assets, R&D-projects, or marketing and sales innovations (to name but a few possible kinds of investment opportunities). For two reasons, this task is however of particular importance when operating in financial markets: First, financial assets often show a (considerably) higher value fluctuation than real investments. That is why volatility timing is of particular importance with regard to this asset class (see *Busse* (1999); *Fleming et al.* (2001); *Fleming et al.* (2003)). Second, financial assets are (at least to a large extent) tradable in the short run. In contrast, a short-term divestiture of real investments is most often accompanied by a considerable loss in value (*Koren/Szeidl* (2002), p. 1) and thus not desirable, even when portfolio risk changes. Therefore, the approach presented in this paper focuses on assets which are traded at financial markets. It is hence particularly relevant to banks, insurance companies, and other financial institutions – or more precisely to the divisions these institutions which are responsible for the proprietary trading of – as their core business consist in realising an efficient asset allocation. To be successful they require an accurate and fast risk re-estimation and asset re-allocation process. Against this background, this paper furthermore focuses on the volatility of financial assets as the probably most widely accepted risk measure for quantifying market risk next to VaR.

There is lot of research dealing with volatility forecasting and particularly addressing the first cause of inaccuracy that volatility of financial assets *permanently changes* over time: *Alexander* (2001), *Figlewski* (2004), and *Poon/ Granger* (2003) – to name but a few – provide a relatively broad overview of conventional methods for forecasting future volatility. In doing so, they distinguish two fundamentally different approaches, namely *time series models* such as linear or exponential moving average models, (Generalised) Autoregressive Conditional Heteroskedasticity ((G)ARCH) models, or stochastic volatility models, and *option based models* such as implied volatility models (see *Bollerslev* (1986, 1987); *Broto/Ruiz* (2004); *Engle* (1982); *Heston* (1993); *Nelson* (1991); *Stoyanov et al.* (2009)). These approaches in particular try to model typical effects regarding future volatility such as persistence, clustering and mean reversion (see *Poon/Granger* (2003)). Furthermore, we can detect an increasing awareness among both researchers and practitioners concerning the drawbacks of different forecasting methods (see e.g. *Andersen/Bollerslev* (1998); *Andersen et al.* (2004); *Barruci/Renò* (2002); *Hentschel* (2003); *Patton/Shepard* (2009); *Poon/Granger* (2005)) as well as the variability of the forecasting results depending on the model used and the market conditions (see *Alexander* (2001), p. 117). Researchers thus recommend to run a plethora of simulations varying both the historical data basis and the scenarios of future market behaviour or even to combine several forecasting methods in order to reach a reasonable volatility estimator (cf. *Yang* (2004)). Although being aware of the significant influence of the forecast horizon and the resultant *time lag* with regard to volatility estimation (*Andersen et al.* (1999); *Poon/Granger* (2003); *Poon/Granger* (2005)), there is only little research addressing this second cause of inaccuracy with regard to future volatility so far. While some authors analyse the value of intra-day data for reaching better volatility estimations (*Andersen et al.* (2003); *Andersen et al.* (2005)), to the best of our knowledge there is no research addressing the field of asset allocation with respect to the simultaneous incorporation of both a predefined risk limit and the time lag between two consecutive volatility estimations. This paper intends to close this research gap and

1. presents an alternative volatility estimator named Risk-at-Risk, which allows for the incorporation of the time lag with respect to volatility forecasting as well as
2. integrates this estimator in the continuous re-allocation of financial assets taking into account in addition a predefined risk limit.

For this purpose, this paper strikes a new path by focusing on the volatility process instead of the return process. In doing so, it simplifies the complex problem of allocating a fixed amount of capital to a large number of various assets by assuming the case of two assets, namely a predefined risky portfolio and a risk-free investment alternative, which allows for building the required capital reserve (see *Yin* (2004) for a similar simplification).

The remainder of the paper is organised as follows: in section 2 we briefly introduce the basics of volatility forecasting as well as the underlying process. In section 3, we present the Risk-at-Risk approach, which allows for the incorporation of the time lag with respect to volatility forecasting. Subsequently, we integrate this Risk-at-Risk in the continuous re-allocation of financial assets additionally taking into account a predefined risk limit. We furthermore demonstrate the approach's practical applicability by means of two examples, one assuming a normally or log-normally distributed future volatility and one using intra-day tick data from Boerse Stuttgart. We finally summarise the central findings and point out limitations and areas for further research in section 4.

2. Forecasting Volatility

The derivation of the Risk-at-Risk approach and its application in the context of asset allocation require a few notations and basic assumptions about the general setting, which are presented in subsection 2.1. In subsection 2.2, we briefly outline common methods of volatility forecasting and explain our incremental contribution. In subsection 2.3, we furthermore discuss the basics of the volatility forecasting process.

2.1. Notations and Basic Assumptions

Focusing on banks, insurance companies, and other financial institutions we can start from the premise that they normally have a planning horizon of several periods $k = 1, \dots, T$, where t_k denotes the beginning of period k. Dealing with financial assets we can furthermore start from the assumptions of classical portfolio theory and assume:

> **Assumption 1:** *Investors have access to a set of risky investment opportunities $i = 1, \dots, n \in \mathbb{N}$, with r_k^i denoting the uncertain return of investment opportunity i during period k, as well as to a risk-free investment alternative, which yields a constant risk-free rate r_f. Each asset is perfectly divisible and traded on a no-frictions market, which is free of transaction costs and taxes. Investment and disinvestment decisions take effect only at the beginning of each period k. For all periods k=1,…,T and all risky investment opportunities $i = 1, \dots, n$, the expected return $\mu(r_k^i)$ is higher than the constant risk-free rate r_f.*

Assumption 2: *Investors are risk-averse and strive for efficient combinations of investment opportunities.*

Assumption 3: *Investors use the standard deviation of return for quantifying their overall risk exposure.*

The standard deviation seems a natural choice for market risk, as it has the same unit as the random variable return and is very closely related to the concept of volatility. We therefore use the terms volatility and standard deviation synonymously and think of the volatility as the annualized standard deviation of percentage change in daily price (see e.g. *Alexander* (2001), p. 5). The presented assumptions characterise the typical setting of classical portfolio theory, where investors aim at maximising a portfolio's expected return subject to a predefined risk constraint in terms of the standard deviation or variance (see e.g. *Markowitz* (1952) or *Sharpe* (1964) for similar assumptions about the market environment). It is, however, important to realize that the approach presented in this paper does not address the problem of determining efficient portfolio weights for various risky assets, but the problem of determining an efficient asset allocation between risky and risk-free assets, which allows for the simultaneous consideration of both a predefined risk limit and the time lag with respect to volatility forecasting. We therefore simplify the complex problem of allocating capital to a large number of various risky assets to the case of two assets, a predefined risky portfolio P, which incorporates some or all of the risky investment opportunities $i = 1,2,\ldots,n$, and the risk-free investment alternative, which serves as a basis for the company's capital reserve. Denoting by $a_k \in [0;1]$ the share of capital that is allocated to risky portfolio P during period k and using common portfolio theory, we obtain the following equations for the overall expected return μ_k and volatility σ_k in period k

$$\mu_k = a_k \cdot \mu_k^P + (1 - a_k) \cdot r_f \text{ and } \sigma_k = a_k \cdot \sigma_k^P \tag{1}$$

where μ_k^P denotes the expected return of the risky portfolio P in period k and σ_k^P its volatility. Regarding these equations, we see that the higher the share a_k of risky invested capital, the larger the overall expected return as well as the overall volatility. On the one hand, investors certainly strive for the highest possible return. On the other hand, σ_k is limited by a predefined risk limit $\bar{\sigma}$. Investors hence have to estimate the future volatility σ_k^P of their risky portfolio P as accurately as possible in order to reach an efficient asset allocation that ensures compliance with this risk limit during the whole period k. We thus briefly present common methods for volatility forecasting in the next subsection.

2.2. Methods for Volatility Forecasting

Following *Alexander* (2001), *Figlewski* (2004), or *Poon/Granger* (2003) – to name but a few – there are two fundamentally different concepts of forecasting volatility: *time series models* and *option based models*. While the latter rely on current market data, the former are based on historical data and require complex mathematical calculation procedures as well as extensive IT support. Compared to option based models time series models very likely show the much bigger time lag problem due to many degrees of freedom with regard to data processing. In the following, we therefore focus on this concept of forecasting volatility. The simplest time series approach is just to take the realized volatility of the last period as an estimator for the volatility of the next period. However, this is also the most error-prone one, at least if there were any extraordinary developments in the last period which cannot be expected to repeat in the next one. That is why so-called mean average models usually do not take only the last but many past volatility realizations. The subgroup of moving average methods focuses on recent data and steadily exchanges the oldest observation for a new one. Exponentially weighted moving averages furthermore put the more weight on an observation the more up to date it is and assume an exponential decline of relevance of older observations. So-called ARCH models in particular aim at accounting for the fact that volatility is characterized by heteroskedasticity (see *Alexander* (1982), p. 71), i.e. a rather high value in one period is accompanied by a rather high value in the next period (see *Engle* (1982); *Engle/Kroner* (1995)). The basic idea of ARCH models is extended by so-called GARCH methods which additionally consider autoregressive terms, i.e. the fact that volatility tends back to a certain mean value in the long run (see *Bollerslev* (1986, 1987); *Barruci/Renò* (2002)). Stochastic volatility models run another path by expressing future volatility as a stochastic process (*Heston* (1993); *Broto/Ruiz* (2004)). They in particular aim at modelling the specific characteristics of volatility such as mean reversion or persistence by assigning certain helpful properties to the stochastic process.

Summing up, there are already many different time series concepts which aim at estimating future volatility on the basis of historical data while simultaneously considering empirically observable characteristics of volatility's long-term behaviour. These models try to forecast future volatility subject to both its history as well as a perception of future market development. Thus, they focus mainly on the first cause of inaccuracy with respect to volatility forecasting. Employing one or several of these models usually results in a distribution of future volatility. The shape of this distribution depends, among other things, on the forecast horizon. The approach presented within this paper starts at this point and proposes a possibility to consider the specific shape of this distribution by simultaneously taking into account the time lag with regard to volatility forecasting, which is implicitly included in this distribution. In doing so, it complements existing methods for volatility forecasting by making a first proposal how to deal with the distribution of future volatility instead of discussing how to generate a reasonable distribution. Regarding the continuous re-estimation of this distribution and asset re-allocation, financial institutions normally have a well-elaborated forecasting process, which is briefly introduced in the next subsection.

2.3. The Process of Volatility Forecasting

Considering the volatility forecasting process, we first have to establish the following notations: The time interval between two consecutive volatility estimations, which results from the fixation of a certain forecast horizon, is named *forecast time frame* ftf. The time interval between starting a specific calculation and reaching the outcome, which results from the restricted capacity of the employed mathematical methods and IT, is called *calculation time frame* ctf. Accordingly, there is a *delta time frame* Δ denoting the time interval between the end of one calculation and the beginning of the next one. Having these notations in mind, for $k = 1 \ldots, T$ the volatility forecasting process consists of the following three basic steps:

1. Starting at time t_k, one first has to determine the length of the forecast time frame ftf_k by deciding on the length of the forecast horizon. In business practice this ranges from one hour to one year (in some cases one has up to five years; see *Poon/Granger* (2005)). As one calculation has to be finished before the next one can start, the following constraint for the starting point t_{k+1} of the next estimation run has to hold:

   $$t_{k+1} \geq t_k + ctf_k \tag{2}$$

   Accordingly, one first has to determine the length of the calculation time frame und subsequently can fix an arbitrary starting point t_{k+1} of the next estimation run that holds equation (2). We thus obtain the following equivalence for the forecast time frame ftf_k at time t_k:

   $$ftf_k = ctf_k + \Delta_k \tag{3}$$

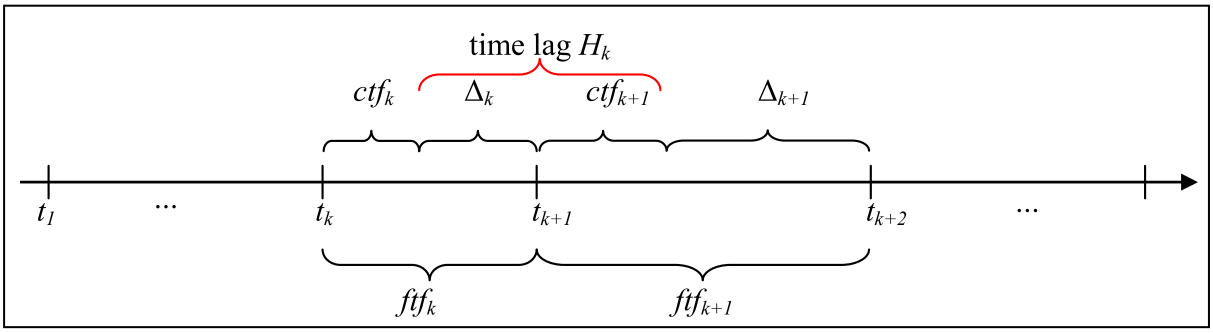
In doing so, the portfolio volatility is implicitly assumed to be time-invariant during the interval $[t_k; t_{k+1})$.

2. Next, one deploys one or several methods for forecasting volatility (see subsection 2.2) in order to estimate future volatility σ_k^P as accurately as possible. These calculations occupy the length of the calculation time frame ctf_k and result in a wide range or an interval of different values, each with a specific probability of occurrence. Subsequently, future volatility σ_k^P is a random variable with a specific distribution ϑ_k^P (cf. *Alexander* (2001), p. 118).

3. At the end of the calculation time frame ctf_k, one uses the determined distribution ϑ_k^P to generate a suitable estimator for σ_k^P. Based on this, one adjusts the asset allocation a_k to reach an overall volatility σ_k somewhere below the predefined risk limit $\bar{\sigma}$. This includes a risk-freely invested capital reserve for ensuring compliance with the risk limit $\bar{\sigma}$ even when volatility varies during Δ_k without the investor noticing it, as he does not run any volatility estimations during Δ_k.

At time t_{k+1}, the end of the forecast time frame ftf_k, the process recurs starting with step 1. Figure 1 illustrates the relevant time frames in detail.

Figure 1: Relevant Time Frames within the Volatility Forecasting Process[1]



Regarding this volatility forecasting process and using equation (3), we have the following time lag H_k between two consecutive estimation runs and the corresponding adjustment of an investor's asset allocation (see also Figure 1):

$$H_k = \Delta_k + ctf_{k+1} \tag{4}$$

The length of this time lag considerably influences the accuracy of volatility estimation (see *Poon/Granger* (2003)). Investors can thus benefit from adjusting it to the effect that, depending on the current market situation, volatility forecasting and asset re-allocation

[1] In Figure 1, the calculation time frame *ctf* and the delta time frame Δ of different periods t_k and t_{k+1} are deliberately charted varyingly long. That is because, in business practice, both time frames are highly variable (see the detailed explanation in the next paragraph).

may take place *in distinct time intervals*. Following equation (4), one has two possibilities for modifying the length of the time lag H_k: varying the calculation time frame ctf_{k+1} or varying the delta time frame Δ_k. However, there are different maturities with respect to the variability of these two parameters. The length of the delta time frame Δ_k can easily be modified in the short-term as it solely depends on the starting point t_{k+1} of the next estimation run. The length of the calculation time frame ctf_{k+1} mainly depends on the applied mathematical methods and IT infrastructure, and can thus hardly be modified in the short-term. It is rather a question of the medium-term IT capital budgeting to invest in a suitable infrastructure in order to form a basis for flexible calculation time frames. Significantly reducing the delta time frame, in an extreme case one can reach the ideal of *almost real-time risk/return management*, where the next volatility forecasting process starts as soon as the preceding one is completed, i.e. $\Delta_k = 0$ and $H_k = ctf_{k+1}$. In doing so, investors can react almost immediately to all relevant, expected and unexpected market movements, as their overall time lag is solely determined by the unavoidable time lag ctf_{k+1} resulting from the restricted capacity of the employed mathematical methods and IT infrastructure. This is of high relevance in times of stormy waters where markets are highly volatile. In contrast, during only slight movements of the underlying markets, a longer re-estimation frequency is usually sufficient to detect relevant changes of volatility. This may help to save expenses for risk management. Against this background, the question arises how to determine a reasonable estimator for σ_k^P which reflects this variability of risk management. We will answer this question in the next section and therefore introduce the notation $\sigma_k^{P,H}$ instead of σ_k^P for indicating that the investor has a time lag of volatility forecasting of length H_k.

3. Asset Allocation Taking into Account the Time Lag of Volatility Forecasting

3.1. The Risk-at-Risk Approach

Having determined the distribution $\vartheta_k^{P,H}$ of the volatility $\sigma_k^{P,H}$ in the course of the volatility forecasting process (see subsection 2.3), the question arises, how to generate a suitable estimator for $\sigma_k^{P,H}$ taking into account the characteristics of this distribution. If one simply takes its expected value, one will lose parts of the available information and the expenditure for generating $\vartheta_k^{P,H}$ would be hardly justifiable (see *Cox et al.* (2008), p. 467). This suggests a more detailed consideration of the distribution's characteristics to reach better volatility estimation. We therefore propose the so-called Risk-at-Risk (RaR) as a basic approach, which allows for the consideration of (parts of) the distribution $\vartheta_k^{P,H}$ when estimating future volatility. As already suggested by its name, this approach is strongly connected to VaR, which is probably the most widespread risk measurement concept for market risk next to volatility in corporate and regulatory practice due to its ease-of-use and simplicity (see *Alexander* (2001) or *Jorion* (1997) for an extensive discussion of VaR and its strengths and limitations). Fixing a target horizon H, where the portfolio is not managed, as well as a confidence level α, $0 \ll \alpha < 1$, we can briefly recap the definition of VaR as:

> Definition 1:
>
> The Value-at-Risk $VaR^{H,\alpha} \in \mathbb{R}$ of a portfolio P over the target horizon H summarizes the maximum future loss of the portfolio's market value over the target horizon H, which is only exceeded with probability $1-\alpha$.

Drawing on the portfolio's volatility $\sigma_k^{P,H}$ instead of its market value, we can transfer the basic idea of VaR to this random variable. Thereby, the target horizon H is canonically given by the corresponding time lag H_k between two consecutive volatility forecasts. Setting a confidence level α, $0 \ll \alpha < 1$, we can thus state (see *Hackenbroch* (2007)):

> Definition 2:
>
> The Risk-at-Risk $RaR^{H,\alpha} \in [0;\infty[$ of a portfolio P over the time lag H_k summarizes the maximum future risk of the portfolio P over the time lag H_k, which is only exceeded with probability $1-\alpha$.

Analogous to the VaR, RaR by design covers the maximum future risk over the given target horizon with the high probability α or, in other words, over the time lag H_k maximum future risk will exceed the RaR only in $(1-\alpha)\cdot 100$ of 100 cases. Following the well-known quantile representation of VaR, the $RaR^{H,\alpha}$ over the time lag H_k at confidence level α can thus be calculated as

$$RaR^{H,\alpha} = q_\alpha^{\vartheta_k^{P,H}} \tag{5}$$

where $q_\alpha^{\vartheta_k^{P,H}}$ denotes the α-quantile of the distribution $\vartheta_k^{P,H}$ of $\sigma_k^{P,H}$.

Determining this estimator for future volatility, one has to specify the confidence level α. In order to ensure compliance with the predefined risk limit no matter how markets develop, for caution's sake it is reasonable to follow general banking practice with regard to VaR determination and choose a rather high value for α such as 98 % to 99 %, (see *Basel Committee on Banking Supervision* (2004), p. 46) as the RaR increases with an increasing value of α (see the identical conclusion for VaR, e.g. *Alexander* (2001), pp. 254). Fixing a rather high value for α makes the RaR a rather conservative estimator for portfolio volatility during the time horizon H_k. Rewriting equation (1), we obtain the following upper limit for the share a_k of capital which can be allocated to the risky portfolio P during period k while simultaneously ensuring compliance with the risk limit $\bar{\sigma}$:

$$a_k \leq \frac{\bar{\sigma}}{RaR^{H,\alpha}} = \frac{\bar{\sigma}}{q_\alpha^{\vartheta_k^{P,H}}} \qquad (6)$$

Allocating a share of up to $\frac{\bar{\sigma}}{RaR^{H,\alpha}}$ of the available capital to the risky portfolio P, the maximum future risk over the time horizon H_k will exceed the predefined risk limit $\bar{\sigma}$ only in $(1-\alpha)\cdot 100$ of 100 cases. Since the higher the share a_k of risky invested capital the higher the overall expected return (see equation (1) in subsection 2.1), investors usually try to exploit this upper limit for a_k as far as possible. Furthermore, there is also empirical evidence for the economic value of volatility timing (see *Busse* (1999), *Fleming et al.* (2001), *Fleming et al.* (2003)). Regarding equation (6), one has two possibilities for increasing this upper limit: one can either increase the overall risk limit $\bar{\sigma}$ or reduce the portfolio's $RaR^{H,\alpha}$. Without limitation the latter can be reached by a reduction of the time lag H_k, as this is accompanied by a reduction of the corresponding $RaR^{H,\alpha}$ (see the identical conclusion for VaR, e.g. *Alexander* (2001), pp. 254). Investors can hence achieve higher returns by increasing the frequency of volatility forecasting and asset re-allocation. In the next subsection, we will demonstrate the approach's practical applicability by means of two examples, one assuming a normally or log-normally distributed $\sigma_k^{P,H}$ and one taking real-world intra-day tick returns of German stock index securities between January and September 2010 from Boerse Stuttgart.

3.2. Exemplary Demonstration of the Risk-at-Risk's Practical Applicability

In order to apply the RaR approach in business practice, investors have to determine the distribution $\vartheta_k^{P,H}$ of their portfolio volatility $\sigma_k^{P,H}$. As volatility shows some characteristic effects such as high persistence, clustering and mean reversion (see *Poon/Granger* (2003)), one normally receives some empirical distribution $\vartheta_k^{P,H}$ for $\sigma_k^{P,H}$. Accordingly, one has to determine the required α-quantile for RaR determination numerically by adding up the relative frequencies of different possible realisations of future volatility. Only if $\sigma_k^{P,H}$ follows some standardised distribution such as a normal, log-normal or exponential one, one is in the happy position of obtaining an analytical solution for the $RaR^{H,\alpha}$. Although being aware that in reality $\sigma_k^{P,H}$ is very rarely normally or log-normally distributed, we nevertheless exemplarily discuss these cases in the following subsection in order to get a basic feel for the RaR.

3.2.1. The Case of Normally or Log-normally Distributed Volatility

If $\sigma_k^{P,H} \sim \nu(E_k^H; S_k^H)$ is normally distributed with expected value E_k^H and standard deviation S_k^H, the following equivalence holds:

$$\sigma_k^{P,H} \sim \nu(E_k^H; S_k^H) \Leftrightarrow \frac{\sigma_k^{P,H} - E_k^H}{S_k^H} \sim \nu(0; 1) \tag{7}$$

where $\nu(0; 1)$ denotes the standardised normal distribution. As the linear transformation of a random variable results in a linear transformation of the corresponding quantiles, we get:

$$Q_\alpha = q_\alpha^{\nu(0;1)} = \frac{q_\alpha^{\nu(E_k^H;S_k^H)} - E_k^H}{S_k^H} \tag{8}$$

where Q_α is the well-known, frequently tabulated α-quantile of $\nu(0; 1)$. We hence obtain:

$$RaR^{H,\alpha} = q_\alpha^{\nu(E_k^H;S_k^H)} = E_k^H + Q_\alpha \cdot S_k^H \tag{9}$$

In practice, the portfolio volatility $\sigma_k^{P,H}$ cannot be assumed to be exactly normally distributed since the normal distribution always adopts negative values with positive probability, while risk measured by the volatility of returns by definition has only non negative values. Therefore, we can either apply a truncated normal distribution, where negative values are assigned by a probability of 0, or use a log-normal distribution $\nu_l(E_k^H; S_k^H)$, since $\nu_l(E_k^H; S_k^H)$ by definition is concentrated on $[0; \infty[$. Assuming $\sigma_k^{P,H}$ to be log-normally distributed with parameters E_k^H and S_k^H, the following equivalence holds:

$$\sigma_k^{P,H} \sim \nu_l(E_k^H; S_k^H) \Leftrightarrow log\, \sigma_k^{P,H} \sim \nu(E_k^H; S_k^H) \Leftrightarrow \frac{log\, \sigma_k^{P,H} - E_k^H}{S_k^H} \sim \nu(0; 1) \tag{10}$$

Analogously we obtain:

$$RaR^{H,\alpha} = q_\alpha^{\nu_l(E_k^H;S_k^H)} = e^{E_k^H + Q_\alpha \cdot S_k^H} \tag{11}$$

Applying these equations we obtain the following upper limits for the share a_k of risky invested capital:

$$a_k \leq \frac{\bar{\sigma}}{E_k^H + Q_\alpha \cdot S_k^H}, \text{ if } \sigma_k^{P,H} \sim \nu(E_k^H; S_k^H) \text{ is normally distributed} \tag{12}$$

and

$$a_k \leq \frac{\bar{\sigma}}{e^{E_k^H + Q_\alpha \cdot S_k^H}}, \text{ if } \sigma_k^{P,H} \sim \nu_l(E_k^H; S_k^H) \text{ is log-normally distributed} \tag{13}$$

Analysing these equations, we see that determinants of the RaR and the resulting upper limit for asset allocation are volatility's distribution parameters E_k^H and S_k^H as well as the α-quantile Q_α of the standardised normal distribution and (for asset allocation) the risk limit $\bar{\sigma}$. Accordingly, RaR increases with increasing values of α (leading to a higher quantile value Q_α), E_k^H, or S_k^H (and vice versa). The upper limit a_k for the share of capital, which can be allocated to the risky portfolio *P*, correspondingly decreases with increasing values of α, E_k^H, or S_k^H, as well as with a decreasing risk limit $\bar{\sigma}$ (and vice versa). The distribution parameters E_k^H and S_k^H, however, essentially depend on the time horizon H_k of volatility forecasting. Accordingly RaR and the upper limit for the asset allocation a_k also depend on the time lag H_k of volatility forecasting.

If portfolio volatility over consecutive periods is additionally assumed to be independent and identically distributed (iid), we can furthermore profitably employ the "square root of time"-rule on the time horizon H_k, as the following equivalence holds (see *Dorfleitner* (2002) or *Steiner/Bruns* (2002)): If volatility $\sigma_k^{P,1}$ over a time lag of length 1 is normally distributed with $\sigma_k^{P,1} \sim \nu(E_k^1; S_k^1)$, where E_k^1 denotes its expected value and S_k^1 its standard deviation, volatility $\sigma_k^{P,H}$ over a time lag of length H_k is also normally distributed with $\sigma_k^{P,H} \sim \nu(E_k^1 \cdot H_k; S_k^1 \cdot \sqrt{H_k})$. To obtain the expected value E_k^H and standard deviation S_k^H over a time lag of length H_k, one thus only has to multiply the expected value E_k^1 over one unit of time by the length of H_k and the standard deviation S_k^1 by the square root of the length of H_k. Using the afore-developed equivalence for the time lag H_k of volatility forecasting (see equation (4)), we thus obtain

- in the case of $\sigma_k^{P,H}$ being independent and normally and identically distributed:

$$RaR^{H,\alpha} = E_k^1 \cdot (\Delta_k + ctf_{k+1}) + Q_\alpha \cdot S_k^1 \cdot \sqrt{\Delta_k + ctf_{k+1}} \tag{14}$$

- in the case of $\sigma_k^{P,H}$ being independent and log-normally and identically distributed:

$$RaR^{H,\alpha} = e^{E_k^1 \cdot (\Delta_k + ctf_{k+1}) + Q_\alpha \cdot S_k^1 \cdot \sqrt{\Delta_k + ctf_{k+1}}} \tag{15}$$

where E_k^1 and S_k^1 denote the distribution parameters over one unit of time. As a result, one gets the following upper limits for the share a_k of riskily invested capital:

$$a_k \leq \frac{\bar{\sigma}}{E_k^1 \cdot (\Delta_k + ctf_{k+1}) + Q_\alpha \cdot S_k^1 \cdot \sqrt{\Delta_k + ctf_{k+1}}} \text{ if } \sigma_k^{P,1} \sim \nu(E_k^1; S_k^1) \quad (16)$$

and

$$a_k \leq \frac{\bar{\sigma}}{e^{E_k^1 \cdot (\Delta_k + ctf_{k+1}) + Q_\alpha \cdot S_k^1 \cdot \sqrt{\Delta_k + ctf_{k+1}}}} \quad \text{if } \sigma_k^{P,1} \sim \nu_l(E_k^1; S_k^1) \quad (17)$$

Analysing these equations, we can derive the following additional results for the RaR in the case of independent and normally or log-normally and identically distributed volatility. Applying the "square root of time"-rule, we get a direct link between the RaR or the upper limit for a_k and the two components Δ_k and ctf_{k+1} of the time lag H_k. RaR increases with increasing values of Δ_k or ctf_{k+1} while a_k decreases with increasing values of Δ_k or ctf_{k+1}. The only parameter that can be influenced directly in the short-term is the delta time frame Δ_k. Depending on the assumption about market developments, investors should use this degree of freedom to increase the overall return by reducing the forecast horizon H_k in times of stormy waters.

Observing volatility in financial markets, one has to notice that future volatility shows some specific characteristics such as high persistence, clustering, and mean reversion (see *Poon/Granger* (2003), *Xiao/Aydemir* (2007)). We therefore have to stress that volatility will not be normally or log-normally distributed in almost all cases in business practice. That is why we present another example for the application of RaR applying real-world intra-day tick data from Boerse Stuttgart in the next subsection.

3.2.2. An Example Applying Intra-Day Data From Boerse Stuttgart

Analysing the logarithmical intra-day tick returns of German stock index shares from Boerse Stuttgart between January and September 2010, we decided to focus on the shares of Linde and Merck, as both show nearly the same trading activity in the regarded period of January 2010 to September 2010 (2638 vs. 2640 sales).

The RaR approach aims at considering the distribution of future portfolio volatility over the time lag of volatility forecasting and provides a volatility estimator for asset allocation subject to a predefined risk limit. Before being able to apply this approach, we thus have to determine a reasonable distribution of future portfolio volatility. We therefore rely on the available data base and estimate a distribution of daily and monthly volatility both separately for each of the shares as well as for a portfolio of 50 % Linde and 50 % Merck shares by proceeding as follows: we first of all build the moving sums of each 14 (for the daily case) or 293 (for the monthly case) intra-day tick returns and thus get 2625 daily or 2346 monthly return observations. This is possible as we have logarithmical intra-day tick returns (see *Dorfleitner* (2002)). Next, we determine the mean daily or monthly return by building the arithmetic average of the 2625 daily or 2346 monthly return observations. In doing so, we get a mean daily return of -0.063 % for Linde, 0.032 % for Merck, and -0.015 % for the portfolio of both as well as a mean monthly return of -1.324 % for Linde, 0.675 % for Merck, and -0.329 % for the portfolio of both. After that, we determine the absolute frequency distribution of the daily or monthly volatility between January 2010 and September 2010 by building the absolute value of the difference between the mean daily or monthly return and each of the 2625 daily or 2346 monthly available observations and counting the absolute frequency of these absolute values. The result can serve as an approximation for the distribution of the daily or monthly volatility. Figures 2 to 7 show the absolute frequency distribution of daily and monthly volatility for Linde, Merck and the portfolio of both.

Figure 2: Absolute Frequency of the Daily Volatility for Linde

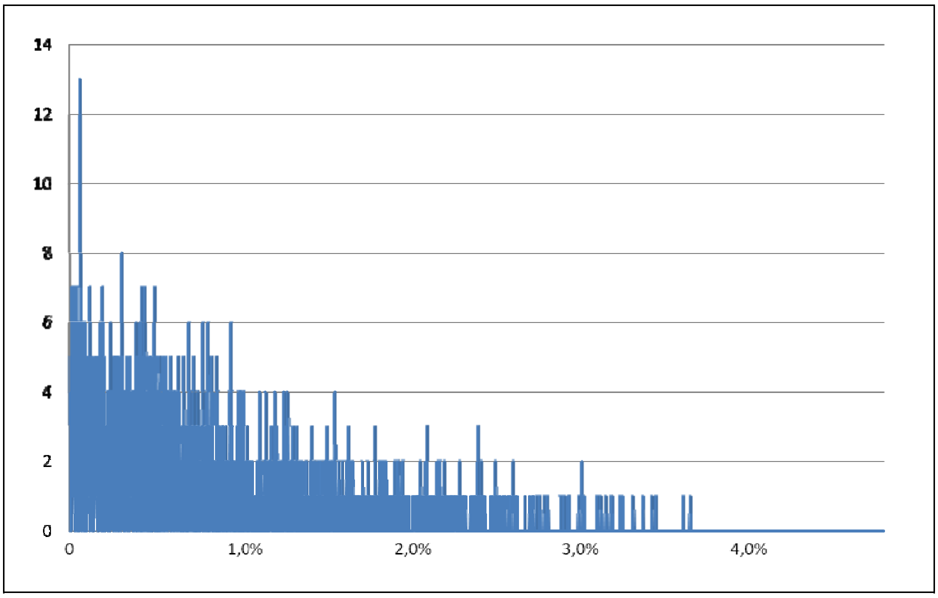


Figure 3: Absolute Frequency of the Daily Volatility for Merck

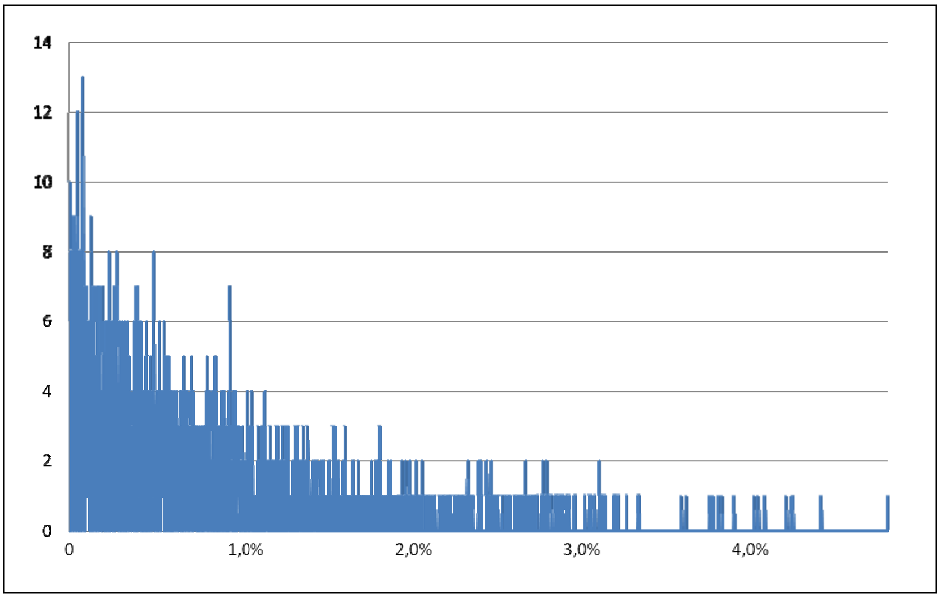


Figure 4: Absolute Frequency of the Daily Volatility for the portfolio of both

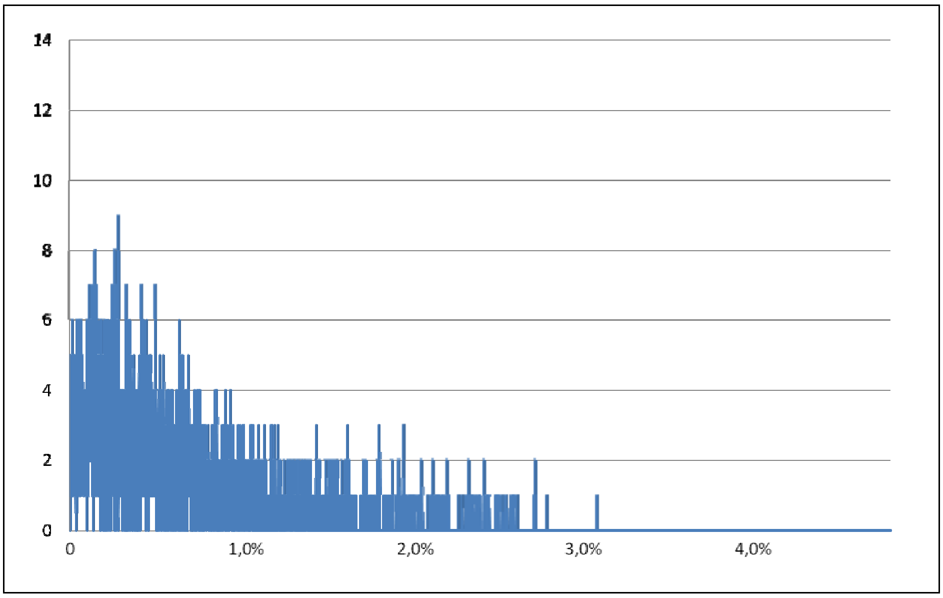


Figure 5: Absolute Frequency of the Monthly Volatility for Linde

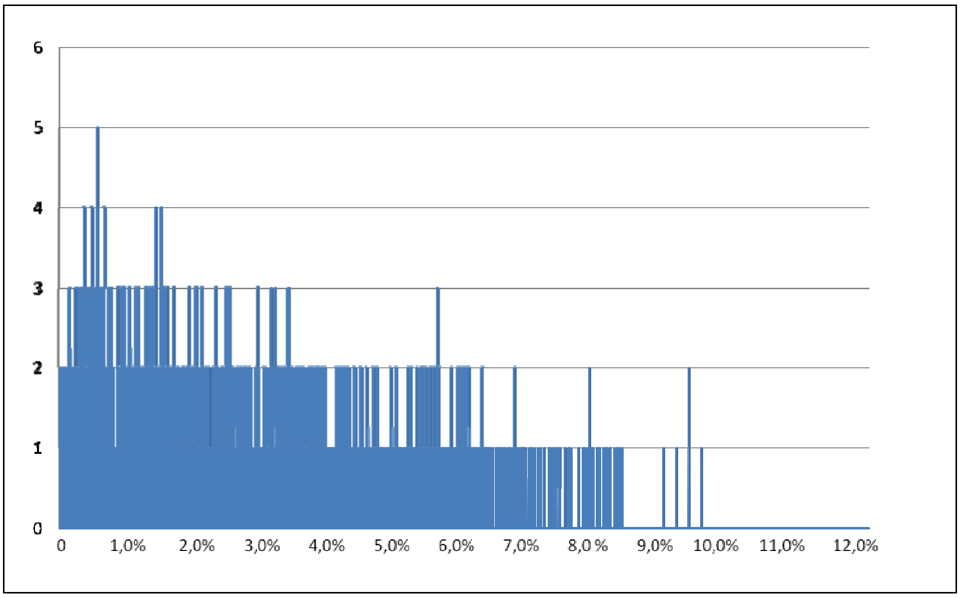


Figure 6: Absolute Frequency of the Monthly Volatility for Merck

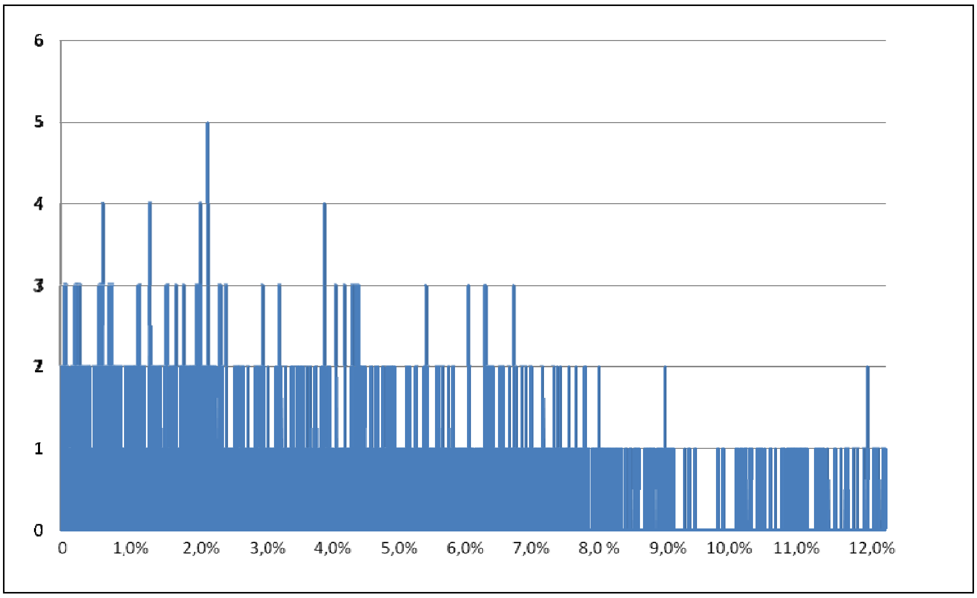
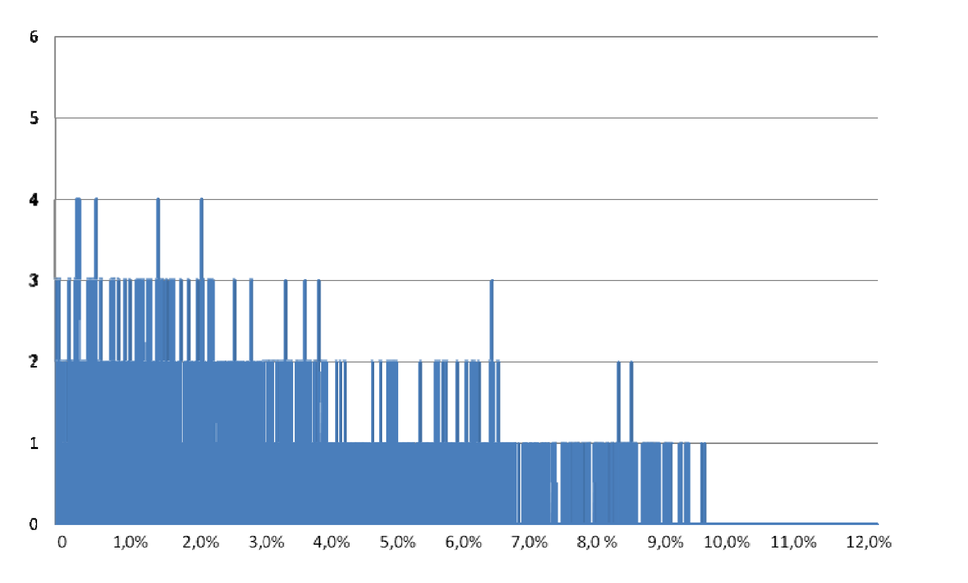


Figure 4: Absolute Frequency of the Monthly Volatility for the portfolio of both



Accordingly, we can determine both the mean daily or monthly volatility (by building the arithmetic average of the 2625 daily or 2346 monthly volatility observations) as well as the 95 % and 99 % quantiles of the identified distribution. Table 1 and 2 provide an overview of the results.

Table 1: Daily volatility: Mean and RaR for Linde, Merck and a portfolio of both values

	Linde	Merck	Portfolio of Linde (50 %) and Merck (50 %)
Mean volatility over one day	0.77 %	0.76 %	0.60 %
	2.12 %	2.20 %	1.63 %
	2.77 %	3.15 %	2.34 %

Table 2: Monthly volatility: Mean and RaR for Linde, Merck and a portfolio of both values

	Linde	Merck	Portfolio of Linde (50 %) and Merck (50 %)
Mean volatility over one month	2.77 %	3.81 %	2.72 %
	6.47 %	8.90 %	6.82 %
	8.23 %	11.37 %	8.85 %

Regarding these tables the following observations are remarkable:

Having a slightly worse performance than the Merck share, the Linde share shows a (considerably) lower daily and monthly volatility. The RaR values follow the inequality $\mathrm{RaR}^{\mathrm{H},95\,\%} \leq \mathrm{RaR}^{\mathrm{H},99\,\%}$ and are furthermore considerably higher than the mean daily or monthly volatility. Applying these volatility estimators in the context of asset allocation, one allocates only one third (if the confidence level is 95 %) to one quarter (if the confidence level is 99%) of the capital to the risky portfolio P compared with using the mean volatility. This is of course accompanied by a profit cut. However, one can simultaneously ensure compliance with a predefined risk limit in 95 % to 99% of all cases and thus guarantee viability and sustainable economic growth.

4. Conclusion and Limitations

Regarding their asset allocation, investors often build higher than necessary risk-freely invested capital reserves to have a buffer against undetected price changes. This usually leads to considerable profit setbacks. One cause for not detecting substantial changes in portfolio volatility is the time lag between two consecutive volatility forecasts, which may lead to the problem that information about the volatility currently being taken is inaccurate as well as significantly outdated. Common methods of volatility forecasting and asset allocation do not consider this time lag so far.

This paper aims at contributing to the closure of this research gap. We therefore present the RaR approach which allows for the incorporation of the time lag between two consecutive volatility forecasts by transferring the basic idea of VaR to portfolio volatility instead of portfolio return. In doing so, we determine the maximum future volatility that is not exceeded during the existing time lag with a predefined confidence level α. Accordingly, RaR is the α-quantile of future portfolio volatility's distribution. We furthermore illustrate how to employ RaR profitably in the determination of an efficient asset allocation between a risky portfolio and a risk-free investment alternative simultaneously taking into account a predefined risk limit, which must not be exceeded. The approach's practical applicability is demonstrated by means of two examples. On the one hand, we determine the RaR and the corresponding upper limit for the share of risky invested capital in the case of a normally and log-normally distributed volatility. We thereby gain the following specific results: determinants for the portfolio's RaR are volatility's distribution parameters expected value and standard deviation as well as the α-quantile of the standardised normal distribution and the time lag between two consecutive risk estimations. The upper limit for the share of risky invested capital is lowered with increasing values of these distribution parameters as well as with increasing values of the confidence level and the time lag between two consecutive risk forecasts or a decreasing risk limit (and vice versa). The only parameter that can be influenced in the short-term is the time lag of volatility forecasting. Adjusting this parameter according to the current market situation in such a way that for instance calculations are performed more often in times of stormy waters enables an economically reasonable asset allocation and an as high return as possible. On the other hand, we determine RaR for daily and monthly volatility of Linde and Merck shares as well as an equally weighted portfolio of both shares on the basis of intra-day tick returns from Boerse Stuttgart for the period of January 2010 to September 2010. Applying these volatility estimators for the determination of the upper limit of the share of riskily invested capital, one will reduce risky investments by two thirds to three quarters, but also ensure to not exceed one's risk bearing ability with an utmost probability.

Besides the discussed advantages of the RaR in the context of volatility estimation and asset allocation, the approach shows some drawbacks that remain to be discussed and reveal directions for further research. First of all, VaR and consequently RaR by design ignore the distribution structure beyond the α-quantile and therefore maybe neglect valuable information about future volatility. As there is empirical evidence that especially the handling of high-frequency data often leads to highly leptokurtic distributions with fat tails (see *Alexander* (2001), p. 82), the consideration of the distribution structure beyond VaR becomes very important. Conditional VaR and similar Expected Shortfall approaches undoubtedly mark the first step towards a better integration of the tails of the distribution and should also be transferred to volatility distribution. Second, the quality of RaR calculation substantially depends on the calculation methods, data and simulation scenarios used for generating the underlying distribution of portfolio volatility. Future research should thus focus on the improvement of existing concepts. Besides, companies cannot expect the portfolio volatility to have an analytical distribution in practice, but have to deal with an empirical one. In contrast to the relatively simple equations for the RaR and the resulting upper limit for the asset allocation presented within this paper for the case of a normally or log-normally distributed volatility, one usually has to estimate the necessary quantiles numerically. Therefore, future research should focus on the identification and further development of appropriate methods for generating a reasonable estimate for the distribution of future portfolio volatility and the necessary α-quantile. Third, there is a trade-off between the benefits of faster volatility calculations and the additional costs caused by the infrastructure that is necessary to perform these calculations. Thus, the guideline "the quicker the better" does not hold in general and future research should study this trade-off in detail, also considering improvements in the field of IT such as service oriented architectures or grid and cloud computing. Finally, both RaR quantification and the corresponding asset allocation are based on mathematical calculations using historical data. However, statistical analysis "made in times of stability is a bad guidance for times of crises" (*Szegoe* (2002), p. 1248) as for instance the turbulences during the last financial and economic crisis show. The lacking robustness of VaR or RaR against extreme market changes also requires future research activity.

Summing up, despite these potentials for improvement, the strict application of the RaR approach can nonetheless provide a basis for an integrated and ideally *real-time risk/return management*, enforcing compliance with a predefined risk limit while simultaneously integrating the (sometimes considerable) time lag with respect to volatility estimation.

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